EXAM GROUP THEORY

April 15, 2005

- Write your name and student number on the first page.
- Start each problem on a new page.
- Answers should be brief, to the point, yet complete.
- Illegible writing will be graded as incorrect.
- Deadline for handing in the exam is Monday, April 25, 5 p.m.

Problem 1 (20 points).

a. Consider the following Cayley table:

$G?$	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	b	e	d	f	c
b	b	e	a	f	c	d
c	c	d	f	a	e	b
d	d	f	c	b	a	e
f	f	c	d	e	b	a

With this product, is the set $\{e, a, b, c, d, f\}$ a group? If so, is there a subgroup? If not, is there a subset which does form a group?

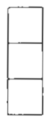
b. Give the elements of the permutation group S_3 in cycle notation. Give the subgroups and their cosets. Find the conjugation classes.

c. For S_3, draw the Young diagrams for the different classes. Draw the corresponding hook tableaux and give the hook products. Draw all the standard Young tableaux and give their Yamanouchi symbols.

d. Discuss the parameter space of SO(3). What changes for SU(2)?

e. What is a Casimir operator? Why are they useful? Give the Casimir operators of SO(3), SU(2), SU(3), and the Lorentz and Poincaré groups. How many Casimir operators do SO(4) and SU(4) have?

f. Determine the dimension of the SU(n) irreps given by:

$[3] =$  , $[21] =$  , $[1^3] =$  , $[2^2] =$  .

To which SU(2) and to which SU(3) irreps do they correspond?

g. Using Young diagrams, prove that for SU(3):

$$\{\mathbf{3}\} \otimes \{\mathbf{3}\} \otimes \{\mathbf{3}\} = \{\mathbf{10}\} \oplus \{\mathbf{8}\} \oplus \{\mathbf{8}\} \oplus \{\mathbf{1}\} \ .$$

Using Young diagrams, reduce for SU(4):

$$\{\mathbf{4}\} \otimes \{\mathbf{4}\} \quad \text{and} \quad \{\mathbf{4}\} \otimes \{\mathbf{4^*}\} \ .$$

Problem 2 (15 points).

Consider the rotation group SO(3). A rotation is given by $R(\boldsymbol{n}) = R(\varphi\hat{\boldsymbol{n}})$, where φ is the rotation angle and $\hat{\boldsymbol{n}}$ a unit vector along the rotation axis.

a. Prove that rotations over the same angle around different axes are conjugate. That is, when $\boldsymbol{a}' = R(\boldsymbol{n})\boldsymbol{a}$, show that

$$R(\boldsymbol{a}') = R(\boldsymbol{n})R(\boldsymbol{a})R^{-1}(\boldsymbol{n}) \ .$$

A finite rotation can be written in terms of Euler angles as

$$D(\varphi, \theta, \chi) = e^{-i\varphi L_z} e^{-i\theta L_y} e^{-i\chi L_z} \ .$$

b. Show that for the irrep labelled by ℓ, the character of the class parametrized by the rotation angle φ is

$$\chi^{(\ell)}(\varphi) = \frac{\sin(2\ell+1)\varphi/2}{\sin\varphi/2} \ .$$

An atom with a valence electron in a *d*-state ($\ell = 2$) is put in a crystal environment with the symmetry of the octahedral group O (we ignore spin). O is the direct (*i.e.* only rotations) symmetry group of the octahedron and the cube; $n(O) = 24$. The incomplete character table of O is:

O	$?\mathcal{E}$	$?C_3$	$3C_4^2$	$6C_2$	$6C_4$
A_1	?	?	?	?	?
A_2	?	?	?	?	-1
E	?	-1	2	?	?
F_1	?	0	?	?	?
F_2	?	?	?	?	?

c. Complete the character table. Explain! In a picture of an octahedron and a cube, draw for each of the classes one of the rotation axes.

d. Calculate the splitting of the five-fold ($\ell = 2$) degenerate energy level. For each resulting level give the remaining degeneracy.

The crystal is subjected to a perturbation with symmetry D_3, with the three-fold axis coinciding with one of those of O.

e. Construct the character table of D_3. Derive which, if any, further splitting of the energy levels is caused by this perturbation.

Problem 3 (15 points).

The Hamiltonian for the isotropic 3D harmonic oscillator is

$$H_0 = p^2/2m + m\,\omega^2 r^2/2 \ ,$$

with $\boldsymbol{r} = (x_1, x_2, x_3)$ and $[x_i, p_j] = i\hbar\,\delta_{ij}$. The energy levels are given by

$$E_{n\ell} = (2n + \ell - 1/2)\,\hbar\omega \ , \qquad n = 1, 2, \ldots \quad \ell = 0, 1, \ldots$$

With N the degeneracy of the levels and P their parity, we get:

$E/\hbar\omega$	$n\ell$	N	P
$3/2$	$1s$	1	+
$5/2$	$1p$	3	−
$7/2$	$2s, 1d$	6	+
$9/2$	$2p, 1f$	10	−
$11/2$	$3s, 2d, 1g$	15	+

etc.

a. Discuss the degeneracy of the energy levels, and compare to what would be expected on the basis of spherical symmetry.

Assuming this degeneracy is not "accidental," we search for a group G, such that every degenerate energy level corresponds to *one* irrep of G.

b. Explain why the group G must have SO(3) as a subgroup.

Define the new variables,

$$\boldsymbol{R} = (X_1, X_2, X_3) = \sqrt{m\omega/\hbar}\,\boldsymbol{r} \quad , \quad \boldsymbol{P} = \boldsymbol{p}/\sqrt{m\hbar\omega} \quad , \quad H_0 = \hbar\omega\, H \ .$$

The generators of SO(3) are $L_i = \varepsilon_{ijk} X_j P_k$. To complete the Lie algebra of G, we have to find additional generators of G that connect the different values of ℓ within the same energy level.

c. Express H in terms of $\boldsymbol{R}$ and $\boldsymbol{P}$ and calculate the commutator $[X_i, P_j]$. Show that the operators $X_i X_j + P_i P_j$ commute with H. Construct a 2nd-rank irreducible SO(3) tensor Q_{ij} that commutes with H.

The set L_i and Q_{ij} is closed under the Lie product, and forms a basis of an 8D Lie algebra.

, SU(3)

d. What is the corresponding well-known eight-parameter Lie group G? Looking back at the table, which irreps of G are realized only? If we include also H in the Lie algebra, it becomes 9D. What is now G? U(3)

only irreps with integer ℓ

e. Find the correspondence between the operators L_i, Q_{ij} and the standard basis of the Lie algebra of the conjectured group G.